Soliton dynamics in a random Toda chain

J. Garnier*
Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 4, France

F. Kh. Abdullaev
Physical-Technical Institute of the Uzbek Academy of Sciences, G. Mavlyanov Street 2-b, 700084, Tashkent, Uzbekistan
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This paper addresses the soliton dynamics in a Toda lattice with a randomly distributed chain of masses. Applying the inverse scattering transform, we derive effective equations for the decay of the soliton amplitude that take into account radiative losses. It is shown that the soliton energy decays as $\sim N^{-3/2}$ for small-amplitude solitons and $\sim \exp(-N)$ for large-amplitude solitons. The decay rate does not depend on the incoming energy for large-amplitude soliton. An important feature is the generation of a soliton gas consisting of a large collection of small solitons (a number of the order of ε^{-2} of solitons with momenta of the order of ε^2, where ε is the strength of fluctuations). The soliton gas plays an important role in that the changes in the conservation equations cannot be correctly understood if the soliton production is neglected. The role of the correlation length of fluctuations on the soliton decay is discussed. It is shown that in the presence of long-range correlation the Toda soliton is not backscattered, but progressively converted into forward-going radiation. The analytical predictions are confirmed by full numerical simulations.

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I. INTRODUCTION

The propagation of nonlinear waves in disordered media was recently the subject of many investigations. Most results concern the dynamics of waves in continuous media. Different scales have been shown to play important roles. One of them is the localization length characterizing the decay law when the nonlinearity is small. The second one is a nonlinear length, which decays as the wave amplitude increases. It plays a fundamental role when the nonlinearity is large. Below some amplitude threshold, the localization length is less than the nonlinear length, so that the exponential decay of wave is observed. Above the amplitude threshold, the decay law is dramatically reduced which proves that nonlinearity can compete with the exponential localization [1,2].

When the nonlinear media are discrete much less is known about the wave dynamics as a new length scale is coming into the problem, the distance between two neighboring sites of the lattice. In a periodic chain of masses, interactions between lattice oscillations can take the form of a resonant sequence, leading to the transfer of the energies of lattice excitations on large distances. Randomness leads to the detuning of the resonances and to the localization of the energy of an excitation on a finite number of sites. The spectrum of normal modes is pure point [3]. This result is valid for 1D (one-dimensional) and 2D lattices and occurs in 3D lattices for disorder with an intensity of fluctuations larger than some critical value [4,5]. Note, however, that very special 1D configurations have been shown to inhibit the Anderson localization (for example, see Ref. [6] for the random dimer model).

For disordered nonlinear discrete media very few analytical results are available. In most of nonlinear discrete systems like discrete nonlinear Schrödinger (DNLS) equation, nonlinear Klein-Gordon lattice, Fermi-Pasta-Ulam chain, moving localized modes are absent due to the Peierls-Nabarro barrier and standing localized modes exist. Recently, the structure of localized modes in disordered nonlinear discrete media has been studied numerically in Refs. [7,8]. In the latter work, it is shown that localized modes with time-periodic dependence can exist in a disordered lattice with one-site random potential. The frequencies of these modes lie inside the linearized spectrum of the random nonlinear discrete Klein-Gordon model and belongs to the fat cantor set. Consequently, it appears that disorder can cooperate with nonlinearity to localize the energy. It is also noted that a continuous path exists from the Anderson localized modes to the nonlinear localized modes in disordered nonlinear crystals. Enhancement of stability and appearance of bifurcations in disordered nonlinear lattices were demonstrated in Ref. [9].

Time-dependent random perturbations in the Ablowitz-Ladik (AL) model were considered in Ref. [10]. The influence of the multiplicative temporal noise on the localized states (discrete breathers) in the DNLS equation is investigated in Ref. [11]. It is shown that the white noise and nonlinear damping will cause the decay of the breather. The intensity decays approximately linearly with time.

Moving localized solutions exist in integrable nonlinear discrete models. Two models admitting moving discrete solitons are important. The first one is the AL model, which is not encountered in physical situations as far as we know. The wave dynamic in some electric transmission inductance-capacitance (LC) lines is modeled by the superposition of DNLS and AL equations with different weights. Taking into account realistic experimental values for the parameters yields that the DNLS component dominates, so that the processes in LC lines cannot be explained by a small deformation of the AL model [12]. The influence of the random on

*FAX: (33) 5 61 55 60 89; Email address: garnier@cict.fr

site potential has been studied in Ref. [13], where the decay law of an AL soliton has been found. For the understanding of the dynamics of discrete solitons in disordered nonlinear lattices the Toda chain with random parameters is much more interesting. The Toda chain is used as a model for the dynamics of biopolymers such as DNA chains [14], LC transmission lines [15], excitations in anharmonic lattices, lattices of optical solitons in fibers [16]. In polymers like DNA, strings in molecules consist of springs and masses. The longitudinal displacements are induced by the van der Waals potential, which can be approximated by the Toda potential.

The random Toda lattice has been numerically studied in Refs. [17,18]. Two kinds of particles with different masses were randomly distributed along an impure segment. For the soliton energy it is found that the dependence of the transmission rate on the segment length n can be fitted quite well by $1/(1+\alpha n^{\beta})$, with $\beta\approx 1.2$ for a wave number equal to 1 in the dimensionless variables. It is also shown that the decay rate as a function of the wave number of the incident soliton can be represented by a Lorentzian function for small wave numbers and tends to a finite value in the large wave number limit.

In nonlinear discrete media numerical simulations show the existence of power type decay of solitons. Such results have been observed for the nontopological kink-type soliton in a disordered anharmonic lattice with nonlinear nearest-neighbor interaction of quartic type, the random Fermi-Pasta-Ulam (FPU) chain [19], and also in the same type of lattices with a disordered harmonic or anharmonic potential [20]. The decay law for the transmission coefficient of the leading soliton $\sim 1/\sqrt{n}$ has been observed.

The first theoretical approach of these phenomena consists in addressing a dilute system of impurities, where the distances between impurities are larger then the soliton width and each interaction between soliton and impurity can be considered as isolated [1,19]. Another approach consists in using the continuum approximation, and studying the stochastically perturbed wave equations. In Ref. [21], this approach has been applied to the random FPU and the decay law $\sim 1/\sqrt{n}$ for the soliton amplitude has been derived. But this approach, considering only broad solitons, uses indeed the mean field theory and neglects radiation phenomena that are important for the long distance propagation in a random chain, so that it is generally questionable for the nonlinear waves in random media [22].

An important peculiarity of the Toda model is the complete integrability allowing the detailed analysis of the discrete soliton dynamics driven by a weak perturbation. In this paper, we shall consider the Toda chain with a segment containing random masses. The length of the segment of the chain with random masses is assumed to be large $\sim 1/\varepsilon^2$, where ε is the perturbation amplitude. Such a system can also be realized in an electric transmission line with a linear inductance L and a nonlinear capacitance $C(V_n)=Q_0/(F_0-V_0+V_n)$, where V is the applied voltage [23,24]. Here, the inductance L plays the role of mass and the segment of line with random variations of L will correspond to the segment with random masses. In comparison with the random AL chain new phenomena are here possible. One of them is the generation of a soliton gas in the Toda chain driven by a random perturbation.

The paper is organized as follows. In Sec. II, we give a review of the homogeneous Toda chain and an introduction to the inverse scattering transform (IST) that is necessary to analyze the long distance evolution of the Toda lattice soliton driven by a random perturbation. In Sec. III, we derive the evolution equations for the scattering data under random perturbations and the equations for the soliton parameters taking into account the radiative losses. We analytically study the decay law of the discrete soliton and we exhibit different regimes depending on the soliton parameter and the correlation length of the medium. Comparisons of the analytical predictions with results of numerical simulations of the random Toda chain are presented in Sec. IV.

II. THE HOMOGENEOUS TODA CHAIN

The model consists of a one-dimensional chain of particles. Each particle with mass one interacts through a nearest-neighbor exponential potential. The difference equation that governs the dynamics of a one-dimensional lattice with exponential interaction of nearest neighbors is deduced from Newton's law [15]

$$\ddot{x}_n=\exp(x_{n+1}-x_n)-\exp(x_n-x_{n-1}), \tag{1}$$

where x_n is the longitudinal displacement of the nth particle from its equilibrium position. The Hamiltonian of this system is

$$\mathcal{H}_0=\sum_{n=-\infty}^{\infty}\frac{1}{2}\dot{x}_n^2+[\exp(x_n-x_{n-1})-(x_n-x_{n-1})-1]. \tag{2}$$

In this section, we give a review and extend the main results reported in Ref. [25].

A. Direct scattering transform

Equation (1) can be rewritten as

$$\dot{c}_n=c_n(v_n-v_{n-1}),\qquad \dot{v}_n=c_{n+1}-c_n,$$

where $c_n=\exp(x_n-x_{n-1})$ and $v_n=\dot{x}_n$. The eigenvalue problem for the continuous spectrum filling the interval $-2\leqslant\lambda\leqslant 2$ reads

$$\sqrt{c_{n+1}}f_{n+1}(k)+\sqrt{c_n}f_{n-1}(k)+v_nf_n(k)=\lambda f_n(k),$$

$$\lambda=k+k^{-1},$$

where k is the spectral parameter that lies in the unit circle $\mathbb{S}^1:=\{k\in\mathbb{C},|k|=1\}$. The Jost functions ψ and ϕ are the eigenfunctions that satisfy the boundary conditions

$$\psi_n(k)\overset{n\to\infty}{\simeq}k^n,\qquad \phi_n(k)\overset{n\to-\infty}{\simeq}k^{-n}.$$

The Jost coefficients are connected to the Jost functions through the identities

$$\phi_n(k)=a(k)\psi_n^*(k)+b(k)\psi_n(k),$$

$$\psi_n(k)=a(k)\phi_n^*(k)-b^*(k)\phi_n(k).$$

The Wronskian of two functions f and g is defined by

$$W(f,g):=\sqrt{c_n}(f_n g_{n-1}-f_{n-1}g_n).$$

Calculating the Wronskian of ϕ and ψ yields

$$W(\phi,\psi)=a(k)W(\psi^*,\psi)=a(k)(k^{-1}-k).$$

Another important point as we shall see in the following is that a admits an analytic continuation inside the unit disk. Finally, symmetry identities hold true

$$a^*(k)=a(1/k),\quad b^*(k)=b(1/k),\quad |a(k)|^2-|b(k)|^2=1.$$

The points on the real axis k_r, $r=1,\dots,R$, $|k_r|<1$, at which $a(k_r)=0$ correspond one to one with eigenvalues of the discrete spectrum. At these points we have

$$\phi_n(k_r)=b_r\psi_n(k_r),\qquad \mathrm{Im}(b_r)=0.$$

Setting $\rho_r=b_r/a'(k_r)$, the set of scattering data $\{a(k),b(k),k\in\mathbb{S}^1,k_r,\rho_r,r=1,\dots,R\}$ is sufficient to reconstruct the Jost functions and the function (x_n).

B. Inverse scattering transform

Given the set of scattering data, set

$$\Omega(n)=-\sum_{r=1}^{R}\rho_r k_r^n+\frac{1}{2i\pi}\oint_{\gamma_u}\frac{b(k)}{a(k)}k^n dk,$$

where γ_u is the positively oriented unit circle. The inverse scattering transform consists in solving the Gel'fand-Levitan-Marchenko (GLM) equation for the kernel K,

$$K(n,m)+\Omega(n+m)+\sum_{l=n}^{\infty}\Omega(l+m+1)K(n,l)=0.$$

Then, the function c_n and v_n are given by

$$c_n=\frac{1-K(n-1,n-2)}{1-K(n,n-1)},\qquad v_n=K(n,n)-K(n-1,n-1).$$

C. Time evolution equation

The time evolutions equations of the scattering data are simple and uncoupled. For any $t\geqslant t_0$,

$$a(k,t)=a(k,t_0),\quad |k|=1,$$

$$b(k,t)=b(k,t_0)\exp[\omega(k)(t-t_0)],\quad |k|=1,$$

$$\rho_r(t)=\rho_r(t_0)\exp[\omega(k_r)(t-t_0)],\quad r=1,\dots,R,$$

where $\omega(k)=k-1/k$.

D. Soliton

The scattering data of a pure soliton are ($q_0>0$, $\epsilon_0=\pm 1$)

$$a(k)=\epsilon_0\frac{k-k_0}{kk_0-1},\qquad b(k)=0,\qquad k_0=\epsilon_0\exp(-q_0), \tag{3}$$

$$\rho_0=\exp[2q_0n_0(t)]\sinh(q_0),$$

$$n_0(t)=n_0(0)-\epsilon_0\sinh(q_0)/q_0t. \tag{4}$$

The corresponding solution is

$$x_n(t)=-\ln\left[1+\frac{e^{q_0}\sinh(q_0)}{\cosh\{q_0[n-n_0(t)]\}}e^{-q_0[n-n_0(t)]}\right]. \tag{5}$$

The soliton momentum and energy are

$$\mathcal{M}_0=2\epsilon_0\sinh(q_0),\qquad \mathcal{H}_0=2[\sinh(q_0)\cosh(q_0)-q_0].$$

Note that the soliton solution is negative valued. Its velocity is negative (positive) if the zero k_0 is positive (negative). The Jost functions for the soliton are

$$\psi_n(k)=k^nA_n^{-1/2}\frac{1-kk_0A_n}{1-kk_0},$$

$$\phi_n(k)=-\epsilon_0k^{-n}A_n^{-1/2}\frac{k-k_0A_n}{1-kk_0},$$

where $A_n=(1+k_0^{2n-2n_0-2})/(1+k_0^{2n-2n_0})$.

E. Conserved quantities

Conserved quantities can be worked out as in any integrable system. The quantities

$$I_0=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\log(c_n),$$

$$I_1=-\sum_{n=-\infty}^{\infty}v_n,\quad I_2=\sum_{n=-\infty}^{\infty}1-c_n-\frac{1}{2}v_n^2$$

are three of the infinite number of conserved quantities for the homogeneous Toda chain. The first integral of motion is proportional to the total displacement $\lim_{n\to+\infty}x_n-\lim_{n\to-\infty}x_n=-I_0/2$. The second integral of motion is proportional to the momentum, $\Sigma_{n=-\infty}^{\infty}\dot{x}_n=-I_1$. Finally, the Hamiltonian (2) can be expressed as a combination of the first and third integrals of motion $\mathcal{H}_0=-I_2+2I_0$. The derivation of the set of conserved quantities is based on the series expansion of the analytic function $a(k)$ as $k\to 0$,

$$\log a(k)\simeq\sum_{j=0}^{\infty}I_jk^j, \tag{6}$$

where the I_j are time independent. The conserved quantities can be expressed in terms of the scattering data. Defining

$n(k) := -\log(1-|b|^2(k)/|a|^2(k))$ for $k\in\mathbb{S}^1$, the integrals of motion I_j can be decomposed into the sums of continuous and discrete parts

$$I_0=\frac{1}{4i\pi}\oint_{\gamma_u}\frac{n(q)}{q}dq+\sum_{r=1}^{R}\log|k_r|, \tag{7}$$

$$I_j=\frac{1}{2i\pi}\oint_{\gamma_u}\frac{n(q)}{q^{j+1}}dq+\frac{1}{j}\sum_{r=1}^{R}k_r^j-k_r^{-j},\qquad j\geqslant 1. \tag{8}$$

III. PROPAGATION WITH AN IMPURE SEGMENT

A. Perturbation model

We assume from now on that the masses of the particles are not equal

$$m_n\ddot{x}_n=[\exp(x_{n+1}-x_n)-\exp(x_n-x_{n-1})], \tag{9}$$

where m_n is the mass of the particle at site n. A finite segment of impure masses is embedded into a homogeneous infinite chain

$$m_n=\begin{cases}1 & \text{for } n\leqslant 0 \text{ and } n\geqslant N^\varepsilon+1\\ 1+\varepsilon V_n & \text{for } 1\leqslant n\leqslant N^\varepsilon,\end{cases}$$

where the small parameter $\varepsilon\in(0,1)$ characterizes the amplitude of the perturbation. $(V_n)_{n\in\mathbf{N}}$ is a chain of identically distributed random variables . They are zero mean $\langle V_n\rangle=0$, they possess finite moments, the chain is stationary so $\langle V_0V_n\rangle=\langle V_mV_{m+n}\rangle$. We may think for instance at the discrete white noise, where the random variables V_n are statistically independent $\langle V_mV_n\rangle=0$ if $m\neq n$ with the variance $\sigma^2=\langle V_n^2\rangle$. This configuration has been studied numerically in Refs. [17,18]. We may also consider a colored noise with a Gaussian autocorrelation function $\langle V_0V_n\rangle=\sigma^2\exp(-n^2/l_c^2)$ with variance σ^2 and correlation length l_c.

The length of the impure segment N^ε is assumed to be large, of the order of ε^{-2}, and we set $N^\varepsilon=[l_0/\varepsilon^2]$. We introduce the slow variable l as $n=[l/\varepsilon^2]$. Here, the brackets stand for the integral part of a real number. We assume that a pure soliton is incoming from the left. The parameter of the soliton is $k_0=-e^{-q_0}$. Note that the total displacement, momentum and Hamiltonian are preserved

$$\mathcal{D}=\lim_{n\to+\infty}x_n-\lim_{n\to-\infty}x_n, \tag{10}$$

$$\mathcal{M}=\sum_{n=-\infty}^{\infty}m_n\dot{x}_n, \tag{11}$$

$$\mathcal{H}=\sum_{n=-\infty}^{\infty}\frac{1}{2m_n}\dot{x}_n^2+[\exp(x_n-x_{n-1})-(x_n-x_{n-1})-1]. \tag{12}$$

B. Evolution of the scattering data

Let us consider a general form

$$\dot{c}_n=c_n(v_n-v_{n-1})+\varepsilon R_n,\qquad \dot{v}_n=c_{n+1}-c_n+\varepsilon S_n.$$

We get the perturbation model (9) from this general form by setting $R_n=0$ and $S_n=V_n(c_{n+1}-c_n)$. In such conditions the Jost coefficients satisfy the coupled equations

$$\frac{da}{dt}=\varepsilon\frac{1}{k-k^{-1}}[\tilde{\gamma}(k)a+\gamma(k)b], \tag{13}$$

$$\frac{db}{dt}=\omega(k)b-\varepsilon\frac{1}{k-k^{-1}}[\gamma^*(k)a+\tilde{\gamma}(k)b], \tag{14}$$

where

$$\gamma(k)=\sum_n \psi_n(k)\left(\frac{R_n}{2c_n^{1/2}}\psi_{n-1}(k)+\frac{R_{n+1}}{2c_{n+1}^{1/2}}\psi_{n+1}(k)+S_n\psi_n(k)\right), \tag{15}$$

$$\tilde{\gamma}(k)=\sum_n \psi_n^*(k)\left(\frac{R_n}{2c_n^{1/2}}\psi_{n-1}(k)+\frac{R_{n+1}}{2c_{n+1}^{1/2}}\psi_{n+1}(k)+S_n\psi_n(k)\right). \tag{16}$$

The presence of the factor $(k-k^{-1})^{-1}$ in Eqs. (13 and 14) is important. It means that a resonance exist close to the value $k=1$, and, thus, small solitons are likely to be generated, as was observed for the random Korteweg-de Vries (KdV) equation [26].

The strategy we shall develop is based on the inverse scattering transform. The random perturbation induces variations of the spectral data. Calculating these changes, we are able to find the effective evolution of the field and calculate the characteristic parameters of the wave. We are interested in the effective dynamics of the soliton propagating through large impure segments with length $N^\varepsilon=[l_0/\varepsilon^2]$. The total energy is conserved but the discrete and continuous components evolve during the propagation. The evolution of the continuous component corresponding to radiation will be found from the evolution equations of the Jost coefficients. The evolutions of the soliton parameter will then be derived from the conservation of the total energy. However, this approach turns out to be a little more tricky than expected because of the generation of new solitons.

C. Convergence of the soliton parameter

We can now state the main result of the paper, whose proof is given in the Appendix. For the statement, we need to define the concept of soliton gas in our framework: A soliton gas is a large collection of small solitons whose total energy is evanescent, while the sum of their momenta is nonzero.

(1) With a probability that goes to 1 as $\varepsilon \to 0$, the wave scattered by a large impure segment with length $[l/\varepsilon^2]$ consists of one main soliton with parameter $q^\varepsilon(l)$, a soliton gas, and radiation.

(2) The process $[q^\varepsilon(l)]_{l\in[0,l_0]}$ converges in probability to the deterministic function $[q_s(l)]_{l\in[0,l_0]}$, which satisfies the ordinary differential equation

$$\frac{dq_s}{dl}=F(q_s), \tag{17}$$

where

$$F(q)=-\frac{1}{4\pi}\int_0^{2\pi}C^2(q,\theta)\hat{R}[2K(q,\theta)]\frac{\sin^2(\theta)}{\sinh^2(q)}d\theta, \tag{18}$$

$C^2(q,\theta)$ is the normalized energy density scattered by the soliton with parameter q per unit distance for a discrete white noise

$$C(q,\theta)=\pi\frac{\sin\left(\theta-\dfrac{q\sin(\theta)}{\sinh(q)}\right)}{\sinh\left(\dfrac{\pi\sin(\theta)}{\sinh(q)}\right)}, \tag{19}$$

$\hat{R}(\kappa)$ is the discrete Fourier transform of the autocorrelation function of $(V_n)_{n\in\mathbf{N}}$,

$$\hat{R}(\kappa)=\sum_{n=-\infty}^{\infty}\langle V(0)V(n)\rangle\cos(\kappa n), \tag{20}$$

$K(q,\theta)$ is the wave number

$$K(q,\theta)=\theta-\frac{q\sin(\theta)}{\sinh(q)}. \tag{21}$$

The first point means that the event "the transmitted wave consists of one soliton plus some other small amplitude wave" occurs with very high probability for small ε, while the second point gives the effective evolution equation of the parameter of the transmitted soliton in the asymptotic framework $\varepsilon \to 0$. Note that $\hat{R}$ is a positive real valued (Wiener-Khintchine theorem). In case of a discrete white noise, $\hat{R}(\kappa)$ is a constant equal to the variance σ^2.

The scattered wave consists of one main soliton (with parameter q_s of order 1), a soliton gas (with quasizero energy but nonzero momentum), and radiation (associated with the continuous spectrum). The induced displacement, momentum, and energy of the radiation are

$$\mathcal{D}_r=-\frac{1}{2\pi}\int_0^l\int_0^{2\pi}C^2(q_s(y),\theta)\hat{R}[2K(q,\theta)]d\theta dy, \tag{22}$$

$$\mathcal{M}_r=\frac{1}{2\pi}\int_0^l\int_0^{2\pi}C^2(q_s(y),\theta)\hat{R}[2K(q,\theta)]\cos(\theta)d\theta dy, \tag{23}$$

$$\mathcal{H}_r=\frac{1}{\pi}\int_0^l\int_0^{2\pi}C^2(q_s(y),\theta)\hat{R}[2K(q,\theta)]\sin^2(\theta)d\theta dy, \tag{24}$$

respectively. We can also compute the induced displacement $\mathcal{D}_g$ and momentum $\mathcal{M}_g$ of the soliton gas since the total displacement and momentum are preserved by the perturbation, so

$$\mathcal{D}_g=2[q_0-q_s(l)]-\mathcal{D}_r, \tag{25}$$

$$\mathcal{M}_g=-2[\sinh(q_0)-\sinh\{q_s(l)\}]-\mathcal{M}_r. \tag{26}$$

The soliton gas actually consists of about ε^{-2} solitons with parameters of order $q_j^\varepsilon \sim \varepsilon^2$. That is why the induced displacement (equal to $2\Sigma_j q_j^\varepsilon$) and the momentum (equal to $2\Sigma_j \varepsilon_j q_j^\varepsilon$) are of order 1, while the energy [equal to $(4/3)\Sigma_j q_j^{\varepsilon 3}$] is of order ε^4 and hence, asymptotically zero. Note that the emission of a soliton gas in an impure segment has been observed in numerical experiments by Kubota [18]. This gas was indeed described as small quasilocalized solitons that escape from the impure segment very slowly.

D. Small-amplitude soliton regime–white noise

If $q \ll 1$, then the scattered energy density can be analyzed more precisely. It is found that the function C is concentrated around $\theta=\pi$ with a bandwidth of the order of q

$$C(q,\pi+qs)\overset{q\ll 1}{\simeq}\frac{2\pi sq}{\sinh(\pi s)}.$$

This means that the radiation is going backward. Integrating establishes that F is simply $F(q)\simeq -4\sigma^2q^3/15$, $q\ll 1$, and Eq. (17) can be solved

$$q_s(l)=\frac{q_0}{\sqrt{1+8\sigma^2q_0^2l/15}}. \tag{27}$$

In terms of energy, the decay rate reads as

$$\mathcal{H}_s(l)=\frac{\mathcal{H}_0}{(1+8\sigma^2q_0^2l/15)^{3/2}}. \tag{28}$$

Note that the decay rate as $l^{-3/2}$ for the soliton energy is in agreement with the numerical simulations carried out in Refs. [17,18]. However, as we shall see in the following section, this decay law is valid only for small-amplitude solitons.

Equation (27) gives the solution of the problem of the soliton propagation in a nonlinear string with a random mass distribution, which has recently attracted the attention of some authors [27]. Indeed, the limit of small amplitude $q_0 \ll 1$ corresponds to the broad Toda soliton occupying many lattice sites. Then, we can obtain for the relative displacement $u_n = x_n - x_{n-1}$ in the continuum limit the equation

$$m(x)u_{tt} = u_{xx} + \frac{1}{2}(u^2)_{xx} + \frac{1}{12}u_{xxxx}, \tag{29}$$

where $m(x) = 1 + \epsilon V(x)$, $V(x)$ is the random function, with $\langle V \rangle = 0, \langle V(x)V(y) \rangle = B(x-y;l_c)$, $B(x-y;l_c) \to 2\delta(x-y)$, $l_c \to 0$. It is the stochastic Boussinesq equation. The unperturbed equation is solvable by the inverse scattering transform. The single soliton solution is

$$u_s(x,t) = \frac{A}{\cosh(\sqrt{A}(x-vt))^2}, \tag{30}$$

where A is the soliton amplitude and the soliton velocity is $v = \pm\sqrt{1+A/3}$. This solution coincides with the Toda solution for the relative displacement for small amplitude with $A = q_0^2, v = 1 + q_0^2/6$. The solution of the linear spectral problem for Eq. (29) is very difficult and no analytical result is known about the soliton evolution in the disordered nonlinear string. But our solution (27) gives for the decay of soliton amplitude in the disordered nonlinear string the result

$$A \approx \frac{A_0}{1 + 8\sigma^2 A_0 x/15}. \tag{31}$$

E. Large-amplitude soliton regime–white noise

The regime when $q_0 \gg 1$ can also be analyzed precisely, as long as $q_s(l) \gg 1$. It is found that the function C becomes independent of θ,

$$C(q,\theta) \overset{q \gg 1}{\simeq} \frac{1}{2}e^q.$$

This means that a broad band radiation is emitted. Integrating establishes that $F(q) \simeq -\sigma^2/4$, $q \gg 1$, so that the decay rate of q_s is linear $q_s(l) = q_0 - \sigma^2 l/4$, which reads as an exponential decay in terms of the energy

$$\mathcal{H}_s(l) = \mathcal{H}_0 \exp\left(-\frac{\sigma^2 l}{2}\right).$$

The energy decay rate is also independent of the energy of the incoming soliton. This remarkable feature was pointed out by the numerical simulations carried out in Ref. [18]. When the value q_s becomes of the order of 1, the decay switches to the power law described in the preceding section.

It should be noted that the decay rate of a large-amplitude soliton is higher than the one of a small-amplitude soliton. This seems in contradiction with previous analysis of solitons driven by random perturbations for other types of integrable systems, such as the NLS equation [1]. However, we cannot extrapolate the results corresponding to the continuous NLS equation to our system for the following two reasons. First, the amplitude and velocity of the NLS soliton are not coupled and in the large mass limit the variations of mass is small, but the variation of velocity is important. In the Toda chain the soliton amplitude and velocity are coupled so the deceleration leads to the damping of amplitude. Second, discreteness plays a primary role when the soliton amplitude is large and the additional scale order of the lattice step comes into the play. The analysis of the randomly perturbed NLS equation has shown that nonlinearity may reduce the exponential localization. The proposed analysis of the randomly perturbed Toda system shows that the interplay between discreteness, nonlinearity, and randomness is more complicated and may lead to an enhanced instability of a large-amplitude soliton. In the large-amplitude regime the soliton width is of the order of one site, and so is the correlation length of the discrete white noise. This involves a strong interaction between the fluctuations of the medium and the soliton. This comment will be confirmed in the following section.

F. Role of the correlation length

Let us assume that $(V_n)_{n\in\mathbf{N}}$ is a colored noise with the autocorrelation function $\langle V_0 V_n \rangle = \sigma^2 \exp(-n^2/l_c^2)$. For an arbitrary l_c the complete expressions (18–20) should be considered. The case $l_c \ll 1$ corresponds to the discrete white noise case. In this section, we shall assume that $l_c \gg 1$ to analyze the soliton dynamics and the influence of long-range correlation.

Let us first consider a small-amplitude soliton $q \ll 1$. The scattered energy density has two peaks at $\theta = 0$ and $\theta = \pi$,

$$\hat{R}[2K(q,\theta)]C^2(q,\theta)|_{\theta=qs} \overset{q\ll 1}{\simeq} \frac{\pi^{5/2}s^2(1+s^2)^2q^6}{36\sinh^2(\pi s)}\sigma^2 l_c \times \exp\left(-\frac{4l_c^2 q^6 s^2(1+s^2)^2}{9}\right),$$

$$\hat{R}[2K(q,\theta)]C^2(q,\theta)|_{\theta=\pi+qs} \overset{q\ll 1}{\simeq} \frac{4\pi^{5/2}s^2q^2}{\sinh^2(\pi s)}\sigma^2 l_c \times \exp(-16\pi^2 l_c^2).$$

The peak around $\theta = \pi$ ($\theta = 0$) corresponds to an emission of backward-going (forward-going) radiation. In the white noise case, we have seen that the peak around $\theta = \pi$ is dominant. In the framework $l_c \gg 1$, the peak around $\theta = 0$ is dominant. This means that, in the presence of long-range correlation, radiation is emitted in the forward direction, and its spectrum is centered around the carrier wave number of the soliton. Integrating the above expressions establishes that

$$F(q) \simeq \begin{cases} -\dfrac{\sqrt{\pi}\sigma^2 l_c q^7}{18\times 35} & \text{if } q^3 l_c \ll 1,\ l_c \gg 1,\ q \ll 1 \\[2ex] -\dfrac{3\sigma^2}{256 l_c^2 q^2} & \text{if } q^3 l_c \gg 1,\ l_c \gg 1,\ q \ll 1, \end{cases}$$

and Eq. (17) can be solved which establishes a power law decay for the soliton energy. The decay rate is found to be maximal when $q^3 l_c \sim 1$. This demonstrates that the strongest

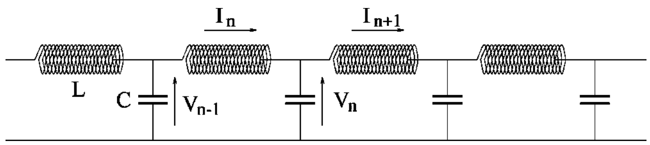

FIG. 1. Electric Toda lattice with linear inductance L and a nonlinear capacitance such as $C(V_n)=Q_0/(F_0-V_0+V_n)$.

interaction between the soliton and the fluctuations of the medium occurs when the width of the soliton is of the order of $l_c^{1/3}$ sites.

Let us consider now a large-amplitude soliton $q\gg 1$. The scattered energy density has a peak at $\theta=0$ with bandwidth l_c^{-1},

$$\hat{R}[2K(q,\theta)]C^2(q,\theta) \overset{q\gg 1}{\simeq} \frac{\sqrt{\pi}}{4}e^{2q}\sigma^2 l_c \exp(-4\theta^2 l_c^2),$$

which means that radiation is only emitted in the forward direction. Integrating yields that $F(q)=-\sigma^2/(64 l_c^2)$, which shows that the total amount of emitted radiation is reduced with respect to the white noise case. This reduction can be interpreted as a result of the fact that the soliton width (of the order of one site) is here very different from $l_c^{1/3}$ (we have $l_c\gg 1$), resulting in a poor interaction. Once again, note that the decay rate of the soliton energy does not depend on the incoming energy.

G. Application to the electric Toda chain

Let us estimate the value of the predicted effect for the possible experiment with the random electric Toda chain. As known, the Toda lattice can be modeled by a special type of electric transmission line (see Fig. 1) with a linear inductance L and a nonlinear capacitance $C(V_n)$. The equation governing this system is [24] $L\ddot{Q}_n=V_{n+1}+V_{n-1}-2V_n$, where Q_n denotes the charge stored in the nth capacitor. Then, we assume that the differential capacitance is modeled by $C(V)=Q_0/(F_0-V_0+V)$, where V is the applied voltage in the region (V_0,V_0+V_n) and $C(V_0)=Q_0/F_0=C_0$. The charge $Q_n(t)$ can be divided into two parts

$$Q_n(t)=\int_0^{V_0}C(V)dV+\int_{V_0}^{V_0+V_n(t)}C(V)dV$$
$$=Q_c+Q_0\ln\left(1+\frac{V_n(t)}{F_0}\right).$$

Since Q_c is constant, we get the equation for the charge

$$L\ddot{Q}_n=F_0\left[\exp\left(\frac{Q_{n+1}-Q_c}{Q_0}\right)+\exp\left(\frac{Q_{n-1}-Q_c}{Q_0}\right)\right.$$
$$\left.-2\exp\left(\frac{Q_n-Q_c}{Q_0}\right)\right].$$

This equation can be rewritten in the form corresponding to the Toda chain equation, as we shall now see. Let us make the electromechanical transformation $(Q_n-Q_c)/Q_0=-r_n$, $LQ_0/F_0=LC_0=m$, where r_n is the relative displacement between two points of mass m. We then get the equation of motion

$$m\ddot{r}_n=2\exp(-r_n)-\exp(-r_{n+1})-\exp(-r_{n-1}).$$

Setting $r_n=x_{n-1}-x_n$, we get the standard form of the Toda chain. If $L=L_0$ is a constant, then a voltage-pulse soliton parametrized by the wave number q_s can be generated

$$V_n(t)=F_0\frac{\sinh^2(q_s)}{\cosh^2[q_s n-\sinh(q_s)t/\sqrt{L_0C_0}]}.$$

Its amplitude is $V_s=F_0\sinh^2(q_s)$ or $\simeq F_0 q_s^2$ for a small-amplitude soliton. In the electric chain L plays the role of mass. Thus, variations of the linear inductance of the form $L(n)=L_0[1+\epsilon(n)]$ with independent random variables $[\epsilon(n)]_{n\in\mathbf{N}}$ corresponds in the mechanical chain to the variations of masses considered in this work. Denoting $\sigma^2=\langle\epsilon(n)^2\rangle$, the amplitude of a small-amplitude voltage-pulse soliton decays as

$$V_s(n)\simeq V_s(0)\left(1+\frac{8\sigma^2V_s(0)n}{15F_0}\right)^{-1},$$

while for a large-amplitude soliton the amplitude decays as

$$V_s(n)\simeq V_s(0)\exp\left(-\frac{\sigma^2 n}{2}\right).$$

In the experiment performed by Hirota [23] the parameters were $L=22\ \mu$H, $C(V)=27V^{-0.48}$ pF, and $V_s(0)\sim 9$–10 V. So for the variations of L equal to $\pm 2\mu H$, we have $\sigma^2\approx 0.01$ and the decay of the large-amplitude soliton by a factor $\sqrt{e}$ at $n=100$.

IV. NUMERICAL SIMULATIONS

The results derived in the preceding section are theoretically valid in the limit case $\varepsilon\to 0$, where the amplitudes of the perturbations go to zero and the size of the random segment goes to infinity. In this section, we aim at showing that the asymptotic behaviors of the soliton can be observed in numerical simulations in the case where ε is small, more precisely smaller than any other characteristic scale of the problem. We integrate Eq. (9) by using a fourth-order Runge-Kutta method.

We consider random masses $m_n=1+\varepsilon V_n$ along the segment $[1,N]$, where V_n are independent random variables with uniform distributions between -1 and 1. We take the value $\varepsilon=0.1$. The simulated evolutions of the soliton energies are presented in Fig. 2 and compared with the theoretical evolutions given by Eq. (17) in the scale $N=[l/\varepsilon^2]$. We can observe, in particular, in Fig. 2(b,c) the algebraic decay of the energy of the soliton, and in Fig. 2(d) the exponential decay of the energy and the cross over to the power law decay. All these results confirm that the effective ordinary differential Eq. (17) describes with accuracy the transmission of a soliton in a randomly perturbed Toda lattice.

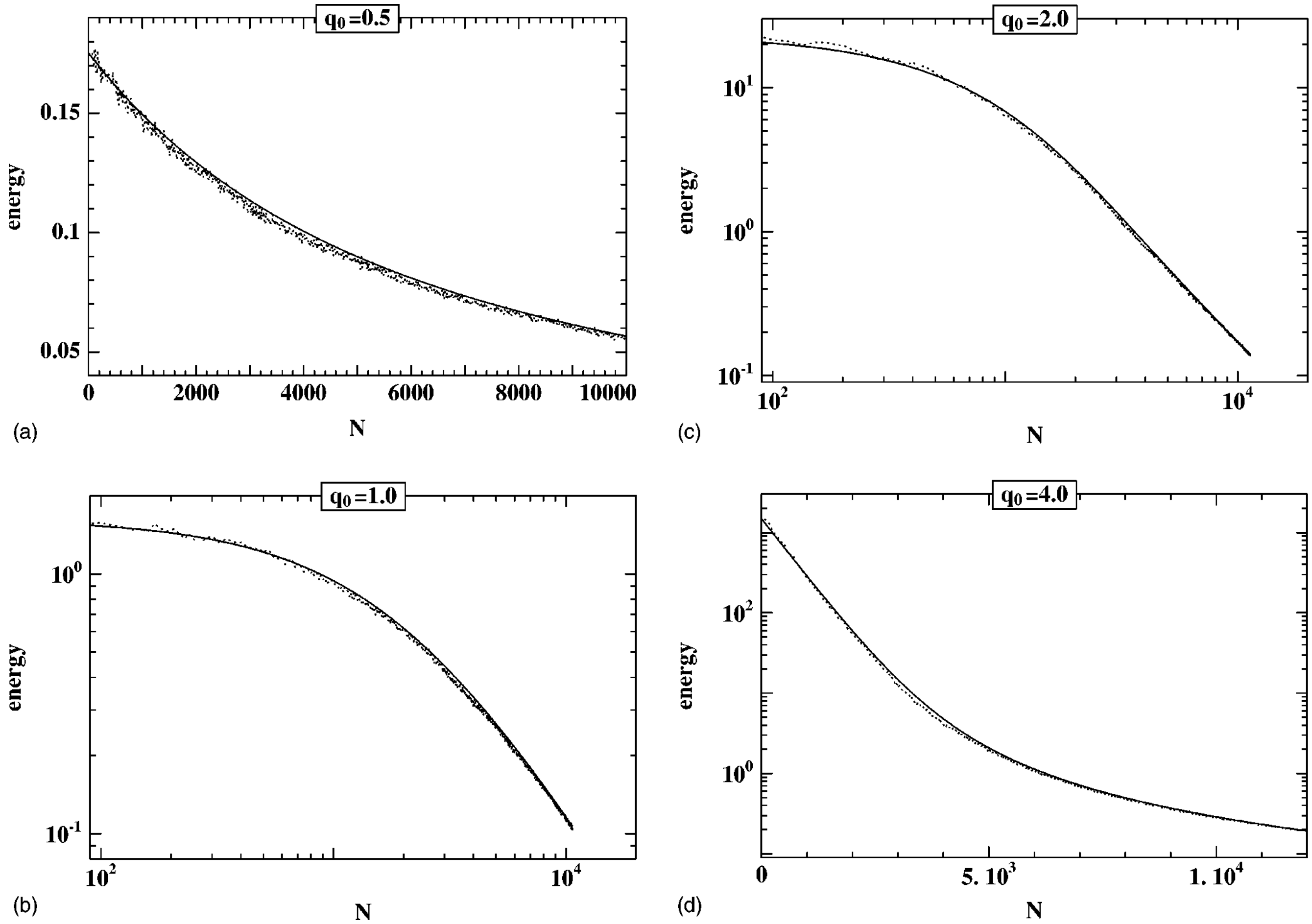

FIG. 2. Energy of the transmitted soliton as a function of the length of the impure segment. Different initial values for the soliton parameters are addressed. The solid lines stand for the theoretical values [Eq. (17)], and the dotted lines plot the results of full numerical simulations. The scales are lin-lin in picture (a), log-log in pictures (b) and (c) (so that the $N^{-3/2}$ decay is noticeable), and lin-log in picture (d) (so that the initial exponential decay is noticeable).

V. CONCLUSION

We have studied the propagation of solitons through a segment of N impure masses. We have shown that the decay rate of the soliton energy is an exponential decaying function of N for large energies, with a decay rate that does not depend on the soliton. The scattering of a large-amplitude soliton is characterized by the emission of a broad band radiation in forward and backward directions. When the energy becomes small, the decay switches to the power law $N^{-3/2}$. The scattering of a small-amplitude soliton is characterized by the emission of a narrow band backward-going radiation. The role of the correlation length of the noise and the influence of long-range correlation have also been discussed. In presence of long-range correlation the soliton dynamics is completely different. The soliton emits a narrow band radiation with a spectrum centered around the soliton wave number, which means that the soliton is not backscattered, but progressively converted into forward-going radiation.

We have put into evidence that the scattering of the soliton generates not only continuous radiation, but also a soliton gas, that is to say a collection of J^{ε}, of order ε^{-2}, solitons with small parameters q_j^{ε}, of order ε^2 (remember ε is the dimensionless parameter that governs the amplitude of the perturbation and the length of the segment $\sim\varepsilon^{-2}$). In the asymptotic framework where ε goes to zero, the soliton gas has nonzero momentum, but quasizero energy. The generation of a large number of small quasilocalized solitons in an impure segment had been observed in numerical simulations by Kubota [18]. These waves were indeed found to escape very slowly from the impure segment. The production of the soliton gas is interesting by itself as a new phenomenon that is not encountered when a random NLS or AL equation is considered, but it should also be pointed out that this production is very important in that we cannot understand correctly the changes in the conservation equations without accounting for soliton production.

APPENDIX A: PROOF OF THE MAIN RESULT

In this section, we outline the main steps of the proof of the result stated in Section III C. The proof follows closely the strategy developed in Ref. [26] in the KdV framework and we shall underline the key points.

In a first time, we carry out the analysis under the so-called adiabatic hypothesis. The adiabatic approximation consists in assuming *a priori* that, while the soliton exists, its evolution and the other components of the wave do not interact. More precisely, we assume that the time evolutions of the Jost coefficients a and b given by Eqs. (13 and 14) depend only on the components of the functions γ and $\tilde{\gamma}$ that are associated with the soliton. We then carry out calculations under this approximation. It reduces the analysis to an infinite-dimensional set of ordinary differential equations with random coefficients, and eventually it provides an expression of the solution (c,v). *A posteriori*, we check for consistency that this approximation is actually justified in the asymptotic framework $\varepsilon \to 0$. More exactly, we show that the components of the functions γ and $\tilde{\gamma}$ which correspond to the interplay between the computed radiation (including the soliton gas) and the soliton, or else which originate from the sole effect of the radiation, can be considered as negligible terms for the soliton evolution.

1. Prove the stability of the zero of the Jost coefficient *a*

The zero corresponds to the soliton. This part strongly relies on the analytical properties of a in the unit disk D_1 of the complex plane. For $k \in D_1$ outside the two small balls with centers at ± 1 and radius $M\varepsilon^2$, we can derive estimates of $a(k)$ by Eqs. (13 and 14). We then apply Rouché's theorem so as to prove that the number of zeros is constant inside the unit disk minus the two small balls. This method is efficient to prove that the main zero (corresponding to the incident soliton) is preserved, but it does not bring control on its precise location in the unit disk. This step is not sufficient to compute the variations of the soliton parameter. Furthermore, it is also possible (and it actually holds true) that new solitons with parameters q_j^ε inside the small balls are generated. Note, however, that the number $J^\varepsilon(t/\varepsilon^2)$ of such new solitons can be bounded above by ε^{-2}. Indeed, we shall see in the next paragraph that the amount $\mathcal{D}_r$ of displacement induced by radiation is of order 1 for propagation distance of order ε^{-2}. The conservation of the total displacement then implies that the displacement $\mathcal{D}_g = 2\sum_{j=1}^{J^\varepsilon} q_j^\varepsilon$ induced by the soliton gas is bounded above by 0 and below by $-\mathcal{D}_r + 2q_0$.

2. Computation of the amount of radiation, and then the variations of the soliton parameters

Under the adiabatic approximation, we solve the evolution Eqs. (13) and (14) so that we get a closed-form expression of the ratio b/a. More exactly, the scattering data $\tilde{b}(k,t) = b/a(k,t)e^{-\omega(k)t}$ at time t_0/ε^2 is given by

$$\tilde{b}\left(k, \frac{t_0}{\varepsilon^2}\right) = -\frac{\varepsilon}{k-k^{-1}} \int_{-\infty}^{t_0/\varepsilon^2} dt\, \gamma^*(k,t) e^{-\omega(k)t},$$

where $\gamma(k,t) = \sum_n V_n \psi_n(k,t)^2 [c_{n+1}(t) - c_n(t)]$. Setting $k = e^{i\theta}$, the increment of

$$\tilde{b}(e^{i\theta},t) = b(e^{i\theta},t)/a(e^{i\theta},t)\exp[-2i\sin(\theta)t]$$

is, up to a phase term

$$\Delta^\varepsilon(\theta) := \tilde{b}\left(e^{i\theta}, \frac{t+\Delta T}{\varepsilon^2}\right) - \tilde{b}\left(e^{i\theta}, \frac{t+\Delta T}{\varepsilon^2}\right)$$

$$= i\varepsilon C(\theta) \sum_{n=N_s(T/\varepsilon^2)}^{N_s[(T+\Delta T)/\varepsilon^2]} V_n \exp[2inK(\theta)],$$

where $N_s(T/\varepsilon^2)$ is the soliton center at time T/ε^2, q_s is the soliton wave number, K is defined by Eq. (21), and

$$C(\theta) = \frac{1}{2\sin(\theta)} \int_{-\infty}^{\infty} [c_{s,1}(t) - c_{s,0}(t)] \times \psi_{s,0}^*(e^{i\theta},t)^2 e^{-2i\sin(\theta)t} dt,$$

$$c_{s,n}(t) = 1 + \frac{\sinh^2(q)}{\cosh^2[qn - q - \sinh(q)t]},$$

$$\psi_{s,0}(e^{i\theta},t) = A(t)^{-1/2} \frac{1 + e^{i\theta - q} A(t)}{1 + e^{i\theta - q}},$$

$$A(t) = \frac{1 + e^{2q + 2\sinh(q)t}}{1 + e^{2\sinh(q)t}}.$$

Computations based on the residue theorem show that C is indeed given by Eq. (19). On the other hand, we can compute the mean and the correlation function of $|\Delta^\varepsilon|^2$ at two nearby frequencies

$$\langle |\Delta^\varepsilon|^2(\theta) \rangle = C^2(\theta) \hat{R}[2K(\theta)] \varepsilon^2 \Delta N_s^\varepsilon,$$

$$|\langle |\Delta^\varepsilon|^2(\theta) |\Delta^\varepsilon|^2(\theta') \rangle| = \langle |\Delta^\varepsilon|^2(\theta) \rangle^2 \left[1 + \mathrm{sinc}^2\left(\frac{[K(\theta) - K(\theta')] \Delta N_s^\varepsilon}{2} \right) + \mathrm{sinc}^2\left(\frac{[K(\theta) + K(\theta')] \Delta N_s^\varepsilon}{2} \right) \right],$$

where $\mathrm{sinc}(s) = \sin(s)/s$ and $\Delta N_s^\varepsilon = N_s[(T+\Delta T)/\varepsilon^2] - N_s(T/\varepsilon^2)$ is of order ε^{-2}. This shows that the frequency correlation radius of the scattered energy density is of order ε^2. This implies that the integrated energy density is a self averaging quantity, so that the total radiated energy is deterministic at first order in ε. The time perturbations preserve the total energy, so we can deduce from the radiated energy the decay of the energy due to the soliton part

$$\mathcal{H}_0 = 2\{\sinh[q^\varepsilon(l)]\cosh[q^\varepsilon(l)] - q^\varepsilon(l)\} + \sum_{j=1}^{J^\varepsilon} 2[\sinh(q_j^\varepsilon)\cosh(q_j^\varepsilon) - q_j^\varepsilon] + \frac{1}{\pi}\int_0^{2\pi} n^\varepsilon(e^{i\theta})\sin^2(\theta)d\theta,$$

where $n^\varepsilon(e^{i\theta}) = -\log[1-|\tilde{b}(e^{i\theta})|^2]$. Since the new generation of solitons consists of $J^\varepsilon = O(\varepsilon^{-2})$ solitons whose energies are of order ε^6, only the discrete energy of the main soliton and the continuous energy of the radiation are of order 1 in the balance of the total energy. This establishes the formula (17).

3. Computation of the form of the scattered wave and check the adiabatic hypothesis

Given the scattering data, we can reconstruct the wave by IST. The procedure is given for instance in Ref. [28] for a general type of perturbed KdV equation. We get the first two terms of the expansion of the kernel $K = K_s + K_r$, where K_s corresponds to the soliton and K_r corresponds to the emitted radiation. By solving the GLM equation, we can deduce the form of the radiation in the vicinity of the soliton as well as the corresponding Jost functions ψ_s and ψ_r. Substituting $\psi = \psi_s + \psi_r$ and $c = c_s + c_r$ into Eqs. (13 and 14) allows us to derive the second-order correction to the Jost coefficient a [of order $\varepsilon^2/(k-k^{-1})$]. This step puts into evidence that new solitons with parameters q_j^ε of order ε^2 are generated.

The final part of the proof consists in checking *a posteriori* the adiabatic hypothesis, that is, to say proving that the radiated wave packet which has been determined here above has actually no noticeable influence on the evolutions (13 and 14) of the Jost coefficients a and b. We must estimate the components of the functions γ and $\tilde{\gamma}$ which have been neglected until now and which are related to the interplay of the main soliton, the soliton gas, and the radiation. These are technical calculations which are based upon the mixing properties of the process V_n. These estimates are qualitatively similar to the ones that are presented in Ref. [29] for the randomly perturbed nonlinear Schrödinger equation. We shall not give the detailed derivations of these estimates because they consist of lengthy calculations that are specific to the perturbation, that is, under consideration. Since this work was performed in Ref. [29] for a randomly perturbed NLS equation, and the technical estimates are essentially similar although the details are different, we thought it better to refer the interested reader to this paper for an example of the technical estimates that are necessary for the check of the adiabatic hypothesis.

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